ON DASGUPTA – HEAL – SOLOW MODEL FOR INVESTMENTS

Codruța Stoica
„Aurel Vlaicu" University of Arad
Department of Mathematics
Bd. Revoluției, No. 81, 310130, Arad
stoicad@rdslink.ro

Abstract. This paper presents the application of the generalized Hartwick rule for investments in the case of Dasgupta-Heal-Solow model. An optimal level of consumption is determined. Also a relation between the decay of the income, the negative investment in manufactured capital and the capital stock is established.

Keywords. Economic growth, investments, the Hotteling rule and the Hartwick rule, the Dasgupta-Heal-Solow model, competitive and efficient economic growth path

1. INTRODUCTION

In a previous paper we have discussed the Solow-Swan neoclassical growth model, where, beside capital stocks and labor, the impact of a third production factor is studied. This idea came from a study on the Hartwick rule and sustainability and on papers due to the researchers Asheim, Buchholz, Withagen, Aronsson, Johansson and Loefgren (as in [1], [2] and [5]). The exhaustible natural resources having its own dynamics lead to a greater influence of the substitution in production. The problem which arouse was if the conventional evaluation of the national income can be modified as to consider the depletion of the natural resources and the depreciation of the quality of environment.

The Hotteling rule and Hartwick rule represents two intertemporal allocation rules in the resources theory. In Dasgupta – Heal – Solow model the first appears as a local efficiency condition and stipulates that along an economic growth path of efficient employment of resources, the price of exhaustible resources grows by a rate equal to the interest rate. Hence a null value of the aggregate net investments leads to a constant consumption. In an economy with production where consumption at any moment t depends not only on the natural capital extraction, but also on the stock of manufactured capital available at moment t is applied the Hartwick rule.

2. THE DASGUPTA-HEAL-SOLOW MODEL

Let us consider an economic model with two aspects of the capital, one of them obtained of natural sources and denoted $n(t)$ at moment t and the other obtained in production activities and denoted $p(t)$ at moment t. The row material obtain from natural resources is denoted $m(t)$. The production function is $y(t) = f\ (p(t),\ m(t))$ and the output $y(t)$ is divided between consumption $c(t)$ and accumulation or investments $i(t) = p'(t)$ at any moment t. Let us suppose that there is no depreciation of the capital, all technological progress is endogenous and the level of population is maintained constant. The competitive intertemporal equilibrium provides market prices for the capital goods.

The properties of the production function are:

(i) function f is homogenous and linear

(ii) function f has positive partial derivatives of first order

$$f'_p = \frac{\partial f(p,m)}{\partial p} > 0 \text{ and } f'_m = \frac{\partial f(p,m)}{\partial m} > 0,\ \forall\ p, m > 0$$

(iii) function f has negative partial derivatives of second order

$$f''_{pp} = \frac{\partial^2 f(p,m)}{\partial p^2} < 0,\ f''_{mm} = \frac{\partial^2 f(p,m)}{\partial m^2} < 0 \text{ and } f''_{pm} = \frac{\partial^2 f(p,m)}{\partial p \partial m} < 0,\ \forall\ p, m > 0$$

(iv) $\lim\limits_{p \to 0} f(p, m) = \infty$ and $\lim\limits_{p \to \infty} f(p, m) = 0,\ \forall\ m > 0$

(v) $f(p(t), m(t)) \geq c(t) + p'(t),\ \forall\ t \geq 0.$

Remark 2.1. From relations (i) and (iv) it is obvious that following statements hold

(a) $\lim\limits_{p \to 0} f'_m(p, m) = 0$ and $\lim\limits_{p \to \infty} f'_m(p, m) = \infty,\ \forall\ m > 0$

(b) $\lim\limits_{m \to 0} f'_p(p, m) = 0$ and $\lim\limits_{m \to \infty} f'_p(p, m) = \infty,\ \forall\ p > 0.$

In this model it is considered a production function of Cobb – Douglas type, defined as $f(p, m) = p^{\alpha} m^{\beta}$ where $\alpha + \beta = 1$, $\beta \in (0, 1)$. This function verifies relations (i) – (v).

3. THE GENERALIZED HARTWICK RULE FOR INVESTMENTS IN DASGUPTA-HEAL-SOLOW MODEL

Definition 3.1. *A pair of functions $(c(t), p(t))$, $t \in [0, \infty)$ such that $c(t) \geq 0$, $p(t) \geq 0$ and $f(p(t), m(t)) \geq c(t) + p'(t)$ for all $m(t) \geq 0$ and $t \geq 0$ is called admissible economic growth path.*

If we denote by $\Omega = \{(c(t), p(t)) \mid t \geq 0\}$ the set of all admissible paths then we consider that Ω is convex and closed and if $(c(t), p(t)) \in \Omega$ then $p(t) \geq 0$ for all $t \geq 0$

The utility function associated to consumption per capita is given by the relation $u(c) = u(c(t))$ and its properties are

(i) function u is differentiable and $\lim\limits_{c \to \infty} u'(c) = 0$

(ii) function u is nondecreasing

(iii) function u is concave.

Let $q(t)$ and $r(t)$ denote respectively the present value prices of the consumption goods and capital stocks at moment t and $\lambda(t)$ denote the utility discount factor at moment t. If we consider $\lambda(t) = \lambda(0)e^{-\delta t}$ then the discount rate is constant, given by

$$\delta = -\frac{\lambda'(t)}{\lambda(t)} = \frac{\lambda(t)}{\int_t^{\infty} \lambda(s) ds}.$$

Definition 3.2. *The admissible economic growth path $(c^*(t), p^*(t))$, $t \in [0, \infty)$ is said to be competitive according to the consumption positive discount factors $\{\lambda(t)\}$, $t \geq 0$ and to the nonnegative discount prices $\{q(t), rq(t)\}$, $t \geq 0$ if for all t following relations hold:*

(i) the instant utility $\lambda(t)u(c) - q(t)c$ is maximized by $c^*(t)$

(ii) $(c^*(t), p^*(t))$ maximizes the income given by $q(t)c + r(t)p' + r'(t)p$ for all admissible paths $(c, p) \in \Omega$.

Following relations hold for $t \geq 0$

$$f(p^*(t), m^*(t)) = c^*(t) + p'^*(t) \text{ where } m^*(t) = p'^*(t) \text{ and } q(t) = r(t)$$

$$r(t) f'_p(p^*(t), m^*(t)) = -r'(t)$$

$$r(t) f'_m(p^*(t), m^*(t)) = 1.$$

Then, as in [2] and [5], the Hotelling rule is verified

$$f'_p(p^*(t), m^*(t))\, f'_m(p^*(t), m^*(t)) = \frac{df'_m(p^*(t), m^*(t))}{dt}$$

and also the generalized Hartwick rule for investments

$$p'^*(t) = f'_m(p^*(t), m^*(t))[\, m^*(t) + i]$$

where i denotes the present constant value of the net investments.

To describe a growth path that satisfies the generalized Hartwick rule for investments we will consider a given consumption level $c^* > 0$ and an initial level of capital due to production and we will determine the quantity of resources used along this path.

Proposition 3.3. *If an admissible path $(c(t), p(t))$, $t \in [0, \infty)$ satisfies the generalized Hartwick rule for investments, the relation $c^*(t) = c^*$ on (t_1,t_2) and for which the relation*

$$f(p^*(t), m^*(t)) \neq c^*, \ \forall t \in (t_1,t_2)$$

hold, then it verifies the Hotelling rule.

Proof. From the hypothesis we obtain

$$f'_p p'^* = f'_m\,(\, m^* + i) = \frac{f''_m}{f'_m}\, p'^*.$$

As $p'^* = f(p^*(t), m^*(t)) - c^* \neq 0$ it follows that the Hotelling rule $f'_p = \dfrac{f''_m}{f'_m}$ is satisfied. □

Lemma 3.4. *If $i \geq 0$ and $c^* > 0$ then for all $p > 0$ there exists one and only one value $m^*(p)$ such the following relation hold*

$$f'_m(p, m(p))[\, m(p) + i] = f(p, m(p)) - c^*.$$

Proof. We consider the function

$$g(p, m) = f'_m(p, m)[\, m + i] - [\, f(p, m) - c^*] = $$
$$= c^* + f'_m(p, m)i - f'_p(p, m)p.$$

According to the properties of the production function obtained in Remark 2.1. statement (b) we obtain that $g(p, m) > 0$ for sufficiently small values of m and $g(p, m) < 0$ for sufficiently large values of m. The continuity property of function $g(\cdot, p)$ implies the existence of at least one value $m^*(p)$ such that $g(m^*(p), p) = 0$. The relation

$$\frac{\partial g(m,p)}{\partial m} = f''_{mm}(p,m)(m+i) < 0$$

implies the existence of a single value $m^*(p)$. □

Lemma 3.5. *Let $c^* > 0$ be a given consumption level, $p(0)$ the initial level of manufactured capital and $i \geq 0$ the positive present constant value of net investments. Then the corresponding path that verifies the generalized Hartwick rule for investments is uniquely determined. Along that path, the investment in production capital is positive at any moment.*

Proof. As the evolution of productive capital stock is given by the relation

$$p'^*(t) = f(p^*(t), m^*(p^*(t))) - c^*$$

where $m^*(p^*(t))$ denotes the resources path at moment t, according to Proposition 3.3 and Lemma 3.4 we obtain by applying the generalized Hartwick rule that $p'^*(t) > 0$ when $i \geq 0$ and $m^*(p^*(t)) \geq 0$. □

Proposition 3.6. *If $c^* > 0$ and $i \geq 0$ then there exists a level $M > 0$ for the natural capital extraction such that*

$$m^*(p) \geq M \text{ for all } p > 0.$$

Proof. The generalized Hartwick rule for investments can also be written as

$$c^* = f(p,\ m^*(p))\left\{1-\frac{f'_e(p,m^*(p))m^*(p)}{f(p,m^*(p))}\left[1+\frac{i}{m^*(p)}\right]\right\}.$$

By hypothesis there exists a constant $r > 0$ such that

$$\frac{f'_e(p,m)m}{f(p,m)} < r$$

and hence it follows that

$$f(p,m^*(p))\left\{1-r\left[1+\frac{i}{m^*(p)}\right]\right\} \geq c^* > 0.$$

If we consider $M = \frac{r}{1-r}$ the proof is completed. □

Let us consider the level $\mu(p)$ of natural resources given by the relation $f(p, \mu(p)) = c$ for all given level of manufactured capital p.

Lemma 3.7. *If $i < 0$ and $c^* > 0$ then there exists one and only one value m^* such that*

$$f'_m(p,\ m^*)(m^* + i) = f(p,\ m) - c^*$$

for all $p > 0$ if and only if $\mu(p) \geq -i$.

Moreover, there exists one and only one value $m^*(p) = -i$ that verifies the relation when $\mu(p) = -i$ and two distinctive values $m^*_1(p) \in (0, -i)$ and $m^*_2(p) > \mu(p)$.

Proof. Let us consider the function

$$g(p,\ m) = c^* + f'_m(p,\ m)i - f'_p(p,\ m)p.$$

As $\frac{\partial g(m,p)}{\partial m} = f''_{mm}(p,\ m)(m + i)$ it follows that $\frac{\partial g(m,p)}{\partial m} > 0$ if $m < -i$ and $\frac{\partial g(m,p)}{\partial m} < 0$ if $m > -i$.

For $m = -i$ we obtain

$$g(p,\ m) = -[f(p,\ m) - c^*] \geq 0$$

if and only if $m < \mu(p)$. Further, the proof is similar with that of Lemma 3.4. □

Lemma 3.8. *Let $c^* > 0$ be a given consumption level, $p(0)$ the initial level of manufactured capital and a negative present constant value of net investments $i \in (-\mu(p(0)), 0)$. Then the corresponding growth path that verifies the generalized Hartwick rule for investments is uniquely determined. Along that path, the investment in production capital is negative at any moment.*

Proof. The evolution of production capital stock is given by the relation

$$p'^*(t) = f(p^*(t),\ m^*(p^*(t))) - c^* = f'_m(p^*(t),\ m^*_1(p^*(t)))[m^*_1(p^*(t)) + i].$$

According to Lemma 3.7 it follows that $m^*_1(p^*(t)) < -i$ *and hence* $p'^*(t) < 0$ *at any moment* t. □

4. SUSTAINABILITY ISSUES

Proposition 4.1. *If $c^* > 0$ and $i > 0$ and $n(0)$ respectively $m(0)$ denote the initial levels of the capital obtained of natural sources and of the capital obtained in production activities then the growth path that verifies the generalized Hartwick rule for investments can not be sustained.*

Proof. According to Lemma 3.5 the extraction of natural resources will tend to ∞ in a finite time interval.

Hence any finite stock of natural capital will be consumed in a finite time interval. □

Remark 4.2. For $i > 0$ the existence of a regular and efficient economic growth path is not possible.

Proposition 4.3. *Let $c^* > 0$, $i \in (-\mu(p(0)), 0)$ and $n(0)$ respectively $p(0)$ denote the initial levels of the capital obtained of natural sources and of the capital obtained in production activities. Then the growth path that verifies the generalized Hartwick rule for investments can not be sustained.*

Proof. By differentiation of the Hartwick rule we obtain

$$(f''_{mp}dmdp + f''_{mm}dmdm^*_1)(m^*_1 + i) + f'_m\, dm^*_1 = f'_p dp + f'_m dm^*_1.$$

Further, according to the properties (ii) and (iii) of the production function f and as $m^*_1(p) + i > 0$ following relations hold

$$\frac{dm^*_1(p)}{dp} = \frac{\dfrac{f'_m}{m^*_1(p)+i} - f''_{mp}}{f''_{mm}} < 0.$$

We obtain $m^*_1(p_1) \geq m^*_1(p_2)$ for $p_1 \leq p_2$. Hence, the output decreases and the negative investment in manufactured capital accelerates and so the stock capital obtained from production will be exhausted in a finite time interval. □

Proposition 4.4. *Let $c^* > 0$, $i \in (-\mu(p(0)), 0)$ and $p(0)$ be the initial level of the capital obtained in production activities. Then there exists an initial level of the capital obtained of natural sources $n(0)$ such that the growth path derived from the path that verifies the generalized Hartwick rule for investments, with negative investments in the manufactured capital until its exhaustion and from a path characterized by a null comsumption, is competitive.*

Proof. We define $n(0) = \int_0^T m^*_1(p(s))ds$ where T denotes the moment when the capital obtained from production is consumed, i.e. $p(T) = 0$. According to Proposition 3.3 it follows that the competitivity conditions are fulfilled. Moreover, both stocks are consumed in a finite time interval. □

Remark 4.5. The economic growth path described in Proposition 4.4 is efficient. Hence the value c^* exceeds strictly the maximal level of consumption that can be sustained for given initial levels of capital stocks $n(0)$ respectively $p(0)$.

5. BIBLIOGRAPHY

[1] Aronsson, T., Johansson, P., Loefgren, K., *A result on investment decisions, future consumption and sustainability under optimal growth*, Working Paper Series in Economics and Finance, No. 49/1995, Stockholm School of Economics

[2] Asheim, G., Buchholz, W., *The Hartwick rule: Myths and facts*, Memorandum, No. 11/2000, University of Oslo

[3] Megan, M., *Controlabilitatea şi optimizarea sistemelor liniare în spaţii Hilbert*, Monografie Matematică, Nr. 4, Tipografia Universităţii din Timişoara, 1975

[4] Megan, M., *Proprietes qualitatives des systemes lineaires controlees dans les espaces de dimension infinie*, Monografie Matematică, Nr. 32, Tipografia Universităţii din Timişoara, 1988

[5] Withagen, C., Asheim, G., Buchholz, W., *On the sustainable program in Solow's model*, Natural Resource Modeling, Volume 16, Number 2, Summer 2003: 219-231